



Wydział Mechaniczny Energetyki i Lotnictwa  
Zakład Wytrzymałości Materiałów i Konstrukcji

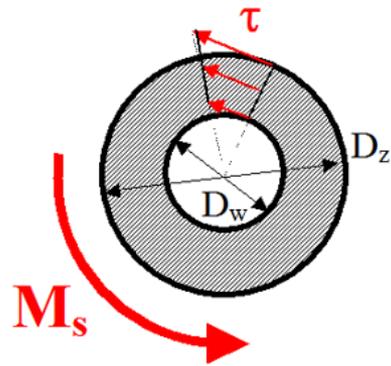


## Ćwiczenia 7

# Pręty skręcane swobodnie

## Przykłady zadań

Rura kołowa



Jednostkowy kąt skręcenia

$$\theta = \frac{M_s}{GJ_0}$$

Naprężenie tnące

$$\tau = \frac{M_s \cdot r}{J_0}$$

Moment bezwładności na skręcanie

$$J_0 = \frac{\pi}{32} (D_z^4 - D_w^4)$$

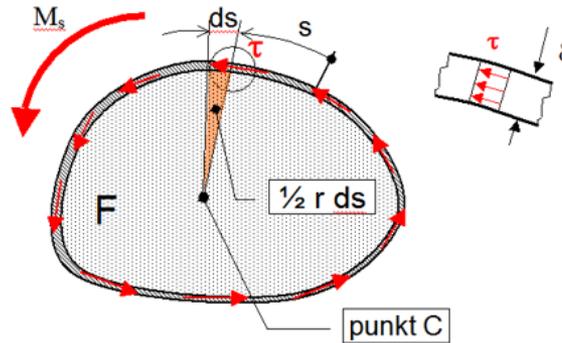
Naprężenie tnące maksymalne

$$\tau_{max} = \frac{M_s}{W_0}$$

Wskaźnik wytrzymałości na skręcanie

$$W_0 = \frac{J_0}{r_{max}}$$

Rura cienkościenna

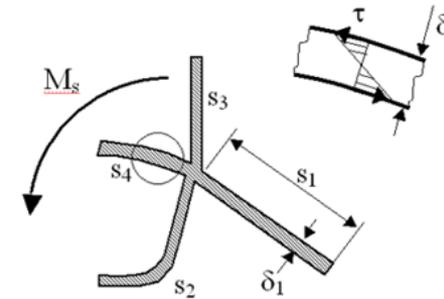


$$\theta(x) = \frac{M_s(x)}{4GF^2} \oint \frac{ds}{\delta(s)}$$

$$\tau = \frac{M_s}{2F \cdot \delta}$$

$$J_s = \frac{4F^2}{\oint \frac{ds}{\delta(s)}}$$

Profil cienkościenny otwarty



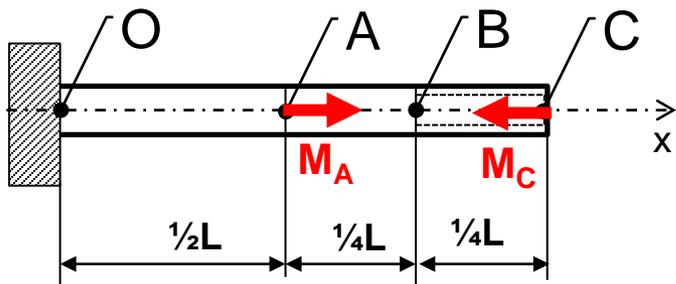
$$\theta = \frac{M_s}{GJ_s}$$

$$\tau = \frac{M_s \cdot \delta_i}{J_s}$$

$$J_s = \frac{1}{3} \sum s_i \delta_i^3$$

Podsumowanie

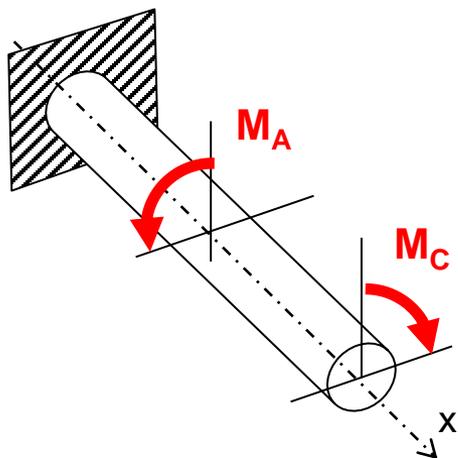
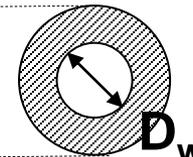
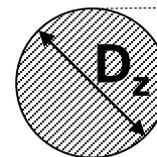
## Zadanie 7.1 Skręcanie pręta o przekroju kołowym



$$\begin{aligned}
 D_z &= 8 \text{ cm} \\
 D_w &= 5 \text{ cm} \\
 L &= 2 \text{ m} \\
 E &= 7 \cdot 10^4 \text{ MPa} \\
 \nu &= 0.35
 \end{aligned}$$

O - B

B - C



$$\begin{aligned}
 M_A &= 3 \cdot M^* \\
 M_C &= M^* = 5 \text{ kNm}
 \end{aligned}$$

### Wyznaczyć:

$M_s(x) = ?$  - rozkład momentu skręcającego

$\tau_{\max}(x) = ?$  - rozkład maksymalnego naprężenia tnącego

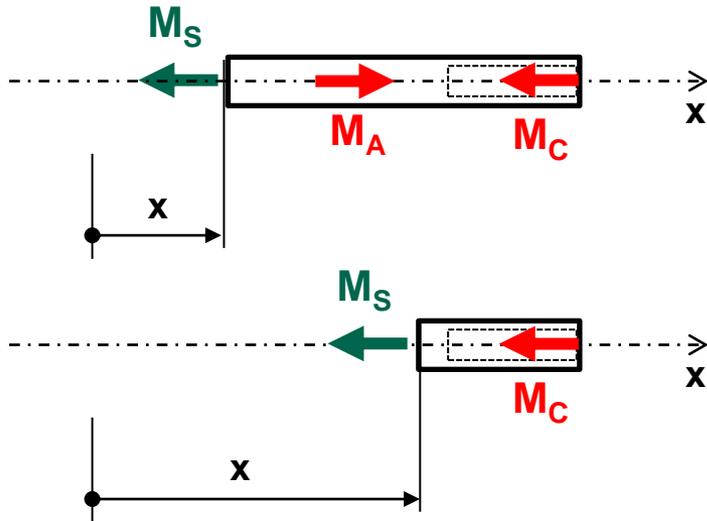
$\theta(x) = ?$  - rozkład jednostkowego kąta skręcenia

$\varphi(x) = ?$  - rozkład kąta skręcenia wzdłuż długości pręta

0. Wyliczenie Modułu Kirchhoffa

$$G = \frac{E}{2(1+\nu)} = 2.6 \cdot 10^4 \text{ MPa}$$

## 1. Rozkłady sił wewnętrznych



Przecinamy myślowo w przedziale O-A

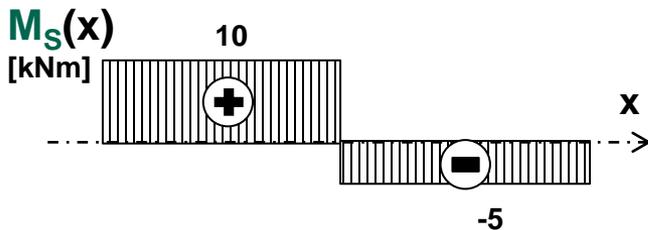
Równanie równowagi momentów względem osi x:

$$-M_S + M_A - M_C = 0 \rightarrow M_S = M_A - M_C = 10 \text{ kNm}$$

Przecinamy myślowo w przedziale A-C

Równanie równowagi momentów względem osi x:

$$-M_S - M_C = 0 \rightarrow M_S = -M_C = -5 \text{ kNm}$$



## 2. Charakterystyki geometryczne przekrojów

Charakterystyki geometryczne przekroju w przedziale O-B

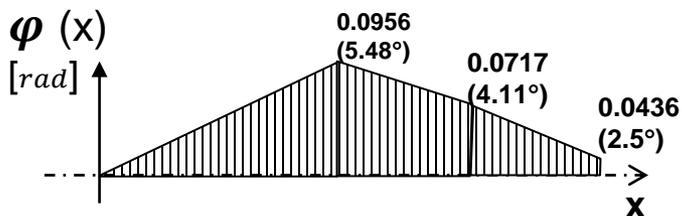
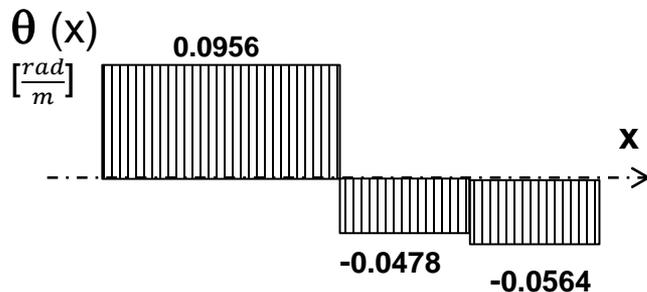
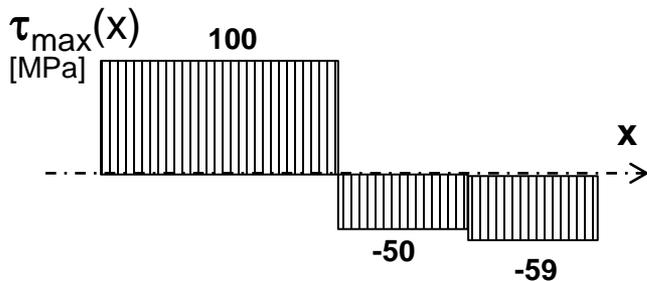
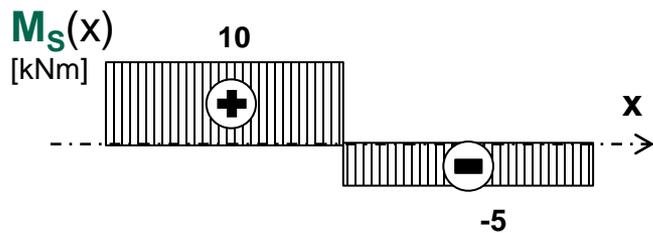
$$J_o = \frac{\pi}{32} D_Z^4 = \frac{\pi}{32} 8^4 \approx 402 \text{ cm}^4 - \text{moment bezwładności na skręcanie}$$

$$w_o = \frac{J_o}{r_{max}} = \frac{402}{4} \approx 100 \text{ cm}^3 - \text{wskaźnik wytrzymałości na skręcanie}$$

Charakterystyki geometryczne przekroju w przedziale B-C

$$J_o = \frac{\pi}{32} (D_Z^4 - D_W^4) = \frac{\pi}{32} (8^4 - 5^4) \approx 340 \text{ cm}^4 - \text{moment bezwładności na skręcanie}$$

$$w_o = \frac{J_o}{r_{max}} = \frac{341}{4} \approx 85 \text{ cm}^3 - \text{wskaźnik wytrzymałości na skręcanie}$$



### 3. Rozkłady maksymalnego naprężenia tnącego

Maksymalne naprężenia tnące dla przekrojów w przedziale O-A

$$\tau_{\max} = \frac{M_S}{W_0} = \frac{10000 \text{ Nm}}{100 \text{ cm}^3} \approx \mathbf{100 \text{ MPa}}$$

Maksymalne naprężenia tnące dla przekrojów w przedziale A-B

$$\tau_{\max} = \frac{M_S}{W_0} = \frac{-5000 \text{ Nm}}{100 \text{ cm}^3} \approx \mathbf{-50 \text{ MPa}}$$

Maksymalne naprężenia tnące dla przekrojów w przedziale B-C

$$\tau_{\max} = \frac{M_S}{W_0} = \frac{-5000 \text{ Nm}}{85 \text{ cm}^3} \approx \mathbf{-59 \text{ MPa}}$$

### 4. Rozkłady jednostkowego kąta skręcenia

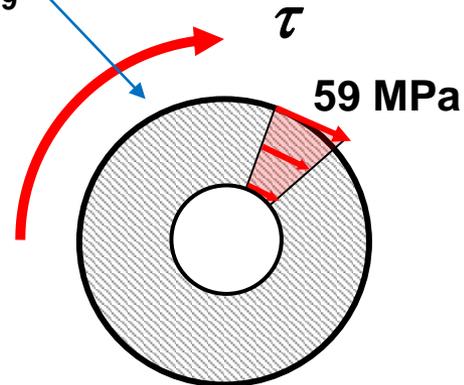
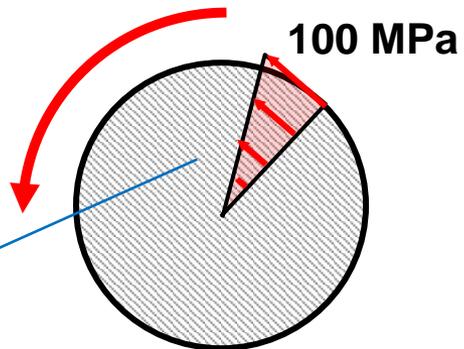
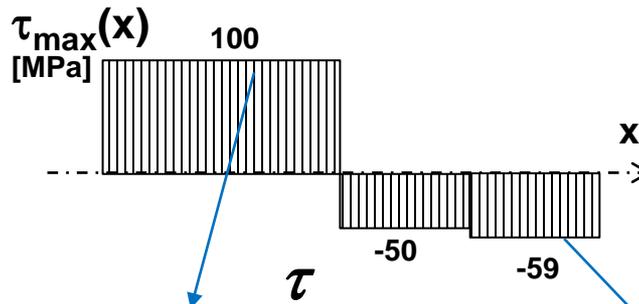
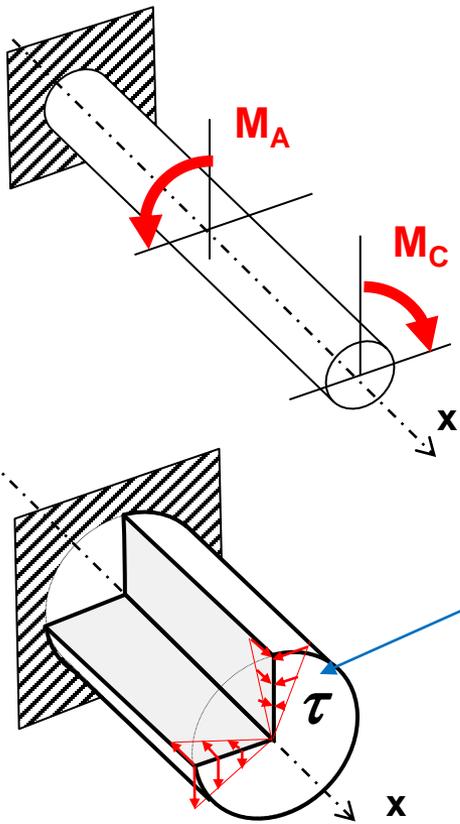
O-A:  $\theta(x) = \frac{M_S(x)}{GJ_0} = \frac{10000 \text{ Nm}}{2.6 \cdot 10^{10} \text{ Pa} \cdot 402 \cdot 10^{-8} \text{ m}^4} \approx \mathbf{0.0956 \frac{rad}{m}}$

A-B:  $\theta(x) = \frac{M_S(x)}{GJ_0} = \frac{-5000 \text{ Nm}}{2.6 \cdot 10^{10} \text{ Pa} \cdot 402 \cdot 10^{-8} \text{ m}^4} \approx \mathbf{-0.0478 \frac{rad}{m}}$

B-C:  $\theta(x) = \frac{M_S(x)}{GJ_0} = \frac{-5000 \text{ Nm}}{2.6 \cdot 10^{10} \text{ Pa} \cdot 341 \cdot 10^{-8} \text{ m}^4} \approx \mathbf{-0.0564 \frac{rad}{m}}$

### 5. Względny kąt skręcenia

$$\varphi(x) = \int_{x_0}^x \theta(x) dx$$



6. Naprężenia zredukowane:

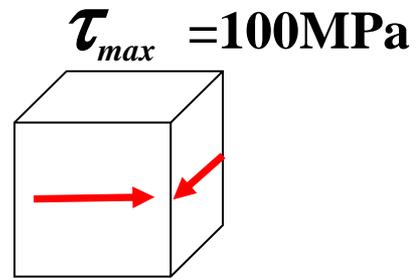
1) Według Hipotezy Treski

$$\sigma_{red}^T = 2 \cdot \tau_{max} = 200 \text{ MPa}$$

2) Według Hipotezy Hubera

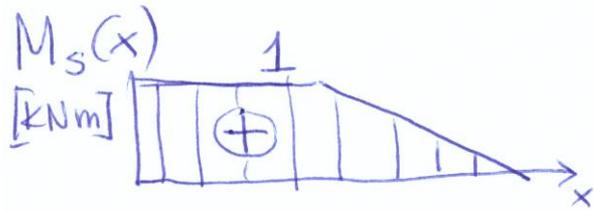
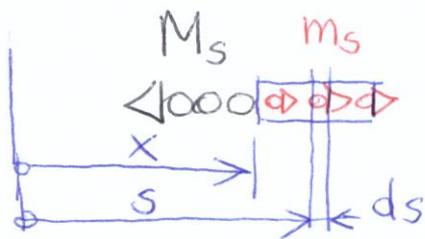
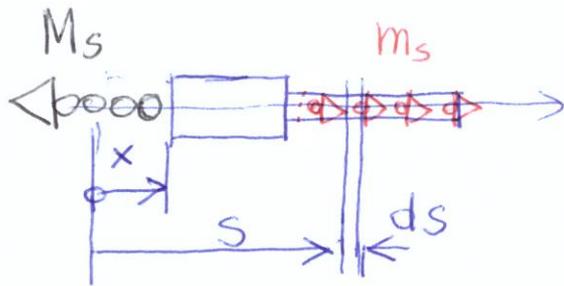
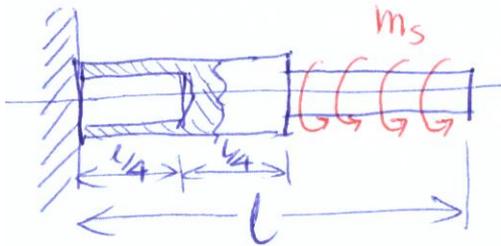
$$\sigma_{red}^{MH} = \sqrt{\frac{1}{2} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

$$\sigma_{red}^H = \sqrt{3} \cdot \tau_{max} = 173 \text{ MPa}$$



## Zadanie 7.2 Skręcanie pręta o przekroju kołowym

Wał turbiny



$$D_z = 50 \text{ mm}$$



$$D_w = 30 \text{ mm}$$

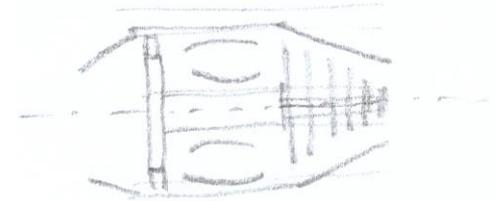
$$d = 30 \text{ mm}$$



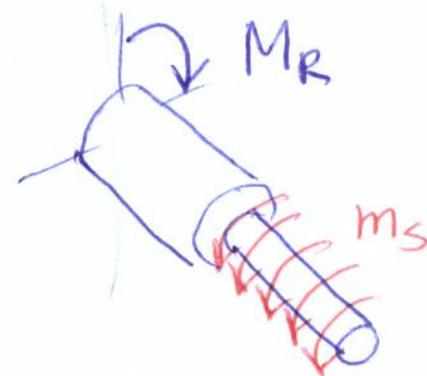
$$l = 1 \text{ m}$$

$$G = 8 \cdot 10^4 \text{ MPa}$$

$$m_s = 2 \frac{\text{kNm}}{\text{m}}$$



silnik turbodołzutowy



r-nie r-gi momentów  $x \in (0, \frac{l}{2})$

$$-M_s + \int_{\frac{l}{2}}^l m_s \cdot ds = 0$$

$$M_s = m_s \cdot s \Big|_{\frac{l}{2}}^l = m_s \cdot \frac{l}{2} = 1 \text{ kNm}$$

r-nie r-gi momentów  $x \in (\frac{l}{2}, l)$

$$-M_s + \int_x^l m_s \cdot ds = 0$$

$$M_s(x) = m_s (l - x)$$

$$M_s\left(\frac{l}{2}\right) = m_s \frac{l}{2} = 1 \text{ kNm}$$

$$M_s(l) = 0$$

Momenty bezwładności na skrajności  
i wskaźniki wytrzymałości na skrajności

$$x \in (0, \frac{l}{4}): \quad J_0 = \frac{\pi (D_z^4 - D_w^4)}{32} = \frac{\pi (5^4 - 3^4)}{32} = 53,4 \text{ cm}^4$$



$$W_0 = \frac{J_0}{r_{\max}} = \frac{53,4}{2,5} = 21,4 \text{ cm}^3$$

$$x \in (\frac{l}{4}, \frac{l}{2}): \quad J_0 = \frac{\pi D_z^4}{32} = \frac{\pi \cdot 5^4}{32} = 61,4 \text{ cm}^4$$



$$W_0 = \frac{J_0}{r_{\max}} = 24,5 \text{ cm}^3$$

$$x \in (\frac{l}{2}, l): \quad J_0 = \frac{\pi d^4}{32} = \frac{\pi \cdot 3^4}{32} = 8 \text{ cm}^4$$



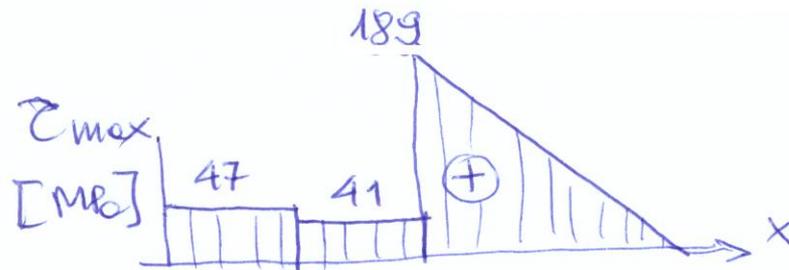
$$W_0 = \frac{J_0}{r_{\max}} = \frac{8}{1,5} = 5,3 \text{ cm}^3$$

## Maksymalne naprężenie tnące

$$x \in (0, \frac{l}{4}): \quad \tau_{\max} = \frac{M_s}{W_0} = \frac{1000 \text{ Nm}}{21,4 \text{ cm}^3} = 47 \text{ MPa}$$

$$x \in (\frac{l}{4}, \frac{l}{2}): \quad \tau_{\max} = \frac{1000}{24,5} = 41 \text{ MPa}$$

$$x \in (\frac{l}{2}, l): \quad \tau_{\max} = \frac{1000}{5,3} = 189 \text{ MPa}$$



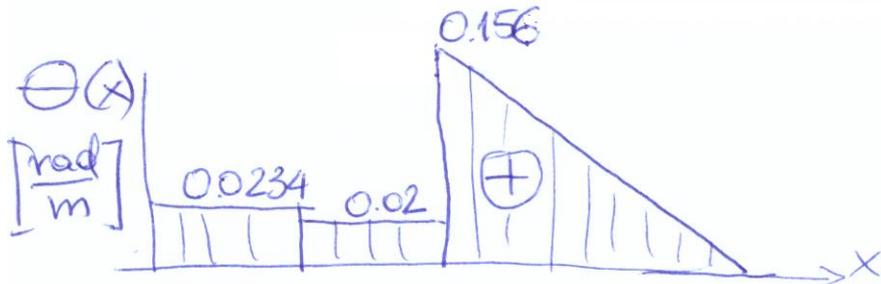
## Jednostkowy kąt skręcenia

$$x \in (0, \frac{l}{4}): \quad \Theta = \frac{M_s}{GJ_0} = \frac{1000 \text{ Nm}}{8 \cdot 10^{10} \frac{\text{N}}{\text{m}^2} \cdot 53,4 \cdot 10^{-8} \text{ m}^4} = 0,0234 \frac{\text{rad}}{\text{m}}$$

$$x \in (\frac{l}{4}, \frac{l}{2}): \quad \Theta = \frac{1000}{8 \cdot 10^{10} \cdot 61,4 \cdot 10^{-8}} = 0,02 \frac{\text{rad}}{\text{m}}$$

$$x \in (\frac{l}{2}, l) \quad \Theta = \frac{m_s(l-x)}{GJ_0} = 0,312(l-x)$$

$$\Theta(\frac{l}{2}) = \frac{1000}{8 \cdot 10^{10} \cdot 8 \cdot 10^{-8}} = 0,156 \frac{\text{rad}}{\text{m}}$$



## Kąt skręcenia

$$x \in (0, \frac{l}{4}): \varphi(x) = \int_0^x \theta(x) dx = 0.0234 \cdot x$$

$$\varphi(\frac{l}{4}) = 0.0234 \cdot 0.25 \text{ m} = 0.0059 \text{ rad} = 0.34^\circ$$

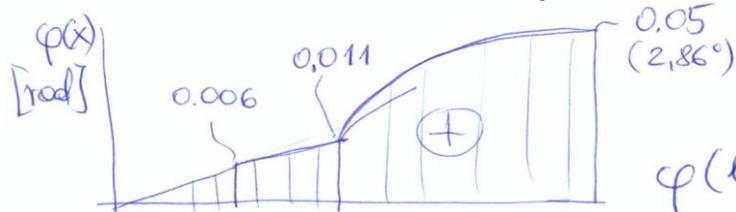
$$x \in (\frac{l}{4}, \frac{l}{2}): \varphi(x) = \varphi(\frac{l}{4}) + \int_{\frac{l}{4}}^x \theta(x) dx = 0.0059 + 0.02(x - \frac{l}{4})$$

$$\varphi(\frac{l}{2}) = 0.0059 + 0.02(0.5 - 0.25) = 0.0109 \text{ rad} (0.62^\circ)$$

$$x \in (\frac{l}{2}, l): \varphi(x) = \varphi(\frac{l}{2}) + \int_{\frac{l}{2}}^x \theta(x) dx = \varphi(\frac{l}{2}) + \int_{\frac{l}{2}}^x 0.312(l-x) dx$$

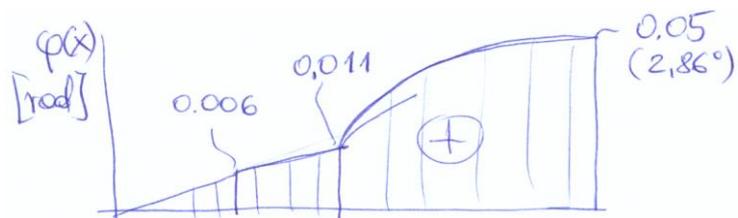
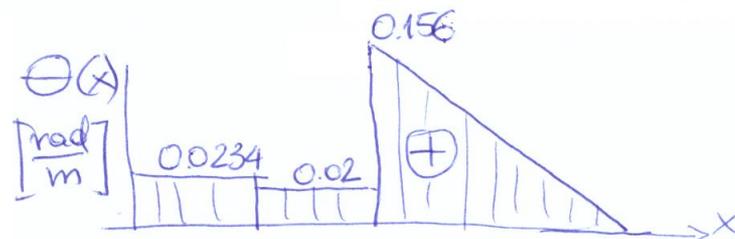
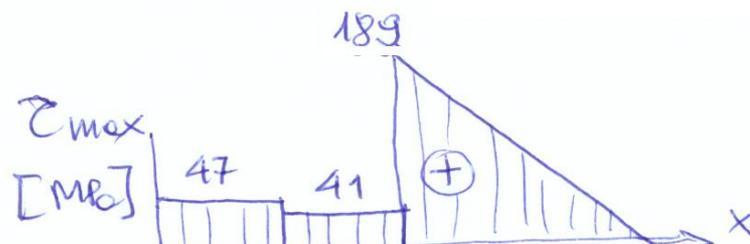
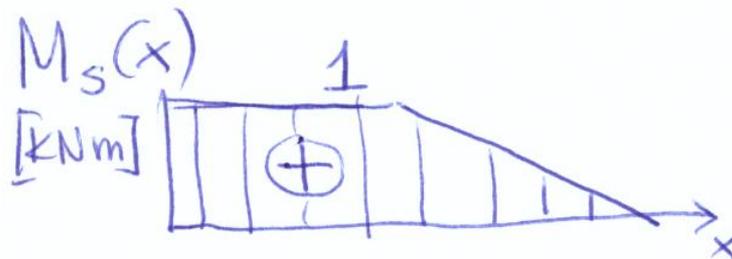
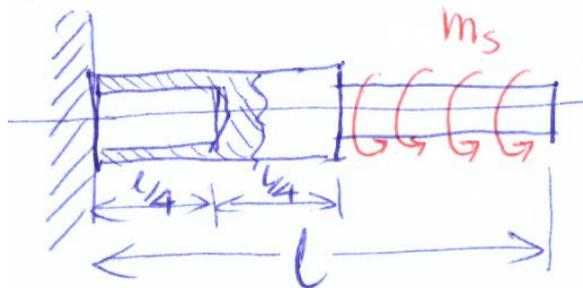
$$\varphi(x) = \varphi(\frac{l}{2}) + 0.312(lx - \frac{1}{2}x^2) \Big|_{\frac{l}{2}}^x$$

$$\varphi(x) = 0.0109 + 0.312(l(x - \frac{l}{2}) - \frac{1}{2}(x^2 - \frac{l^2}{2}))$$



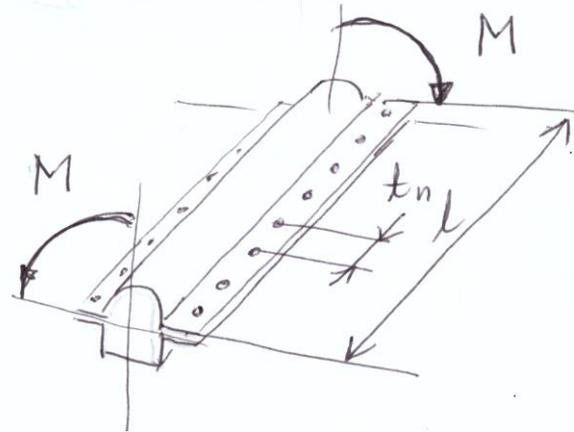
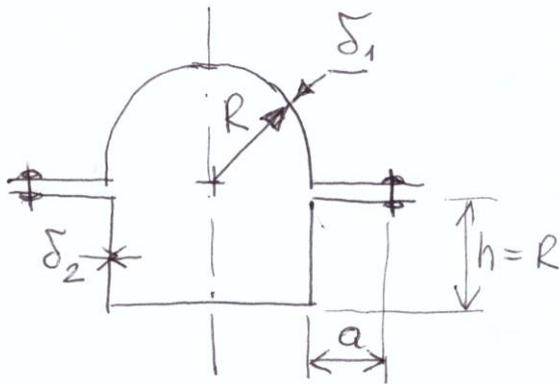
$$\varphi(l) = 0.0109 + 0.312(\frac{l^2}{2} - \frac{3}{8}l^2) = 0.05 \text{ rad} (2.86^\circ)$$

Podsumowanie  
wyników zad. 7.2



## Zadanie 7.3 Skręcanie rury cienkościennej

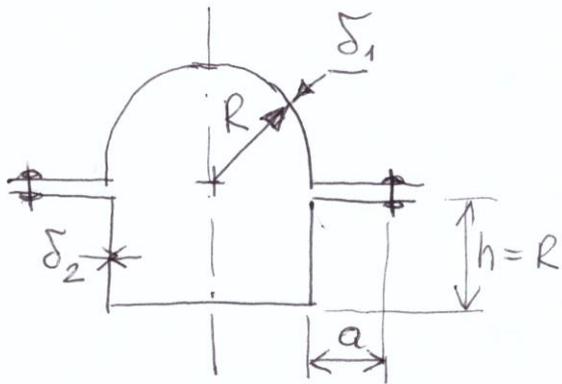
Rura cienkościenna



$$l = 1 \text{ m}, \quad R = 100 \text{ mm}, \quad a = 20 \text{ mm}, \quad \delta_1 = 1.5, \quad \delta_2 = 2$$
$$t_n = 20, \quad M = 10 \text{ kNm}, \quad G = 2,6 \cdot 10^4 \text{ MPa}$$

$$\tau_{\max} = ? \quad \theta = ? \quad T_n = ?$$

Rura cienkościenna

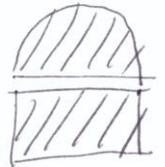


I wzór Bredte

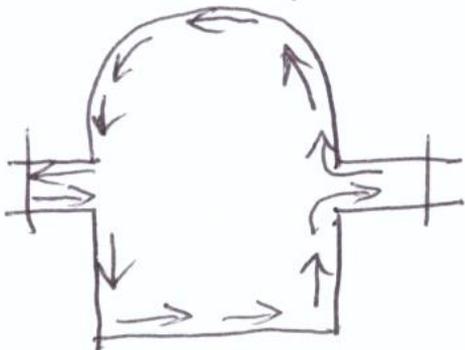
$$\tau = \frac{M_s}{2F\delta}$$

$$F = \frac{1}{2} \pi R^2 + 2R \cdot R$$

$$F = 2R^2 \left(1 + \frac{\pi}{4}\right) = 0.0357 \text{ m}^2$$



$$\tau_1 = 93$$



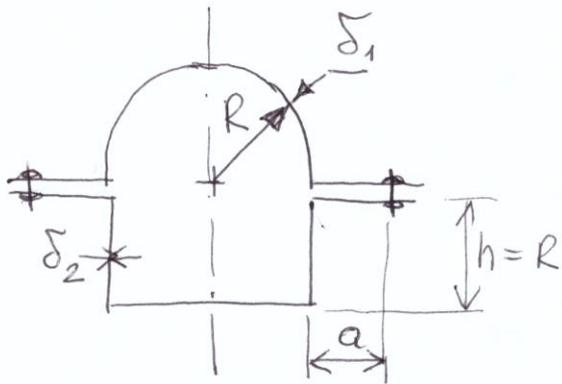
$$\tau_2 = 70$$

$$\tau_1 = \frac{M_s}{2F\delta_1} = \frac{10 \cdot 10^3}{2 \cdot 0.0357 \cdot 1.5 \cdot 10^{-3}}$$

$$\tau_1 = 93 \cdot 10^6 \frac{\text{N}}{\text{m}^2} = \underline{93 \text{ MPa}}$$

$$\tau_2 = \frac{M_s}{2F\delta_2} = \underline{70 \text{ MPa}}$$

Rura cienkościenna



II wzór Bredta  $\Theta = \frac{M_s}{4F^2G} \oint \frac{ds}{\delta}$

$$\Theta = \frac{M_s}{4F^2G} \left( \int_L \frac{ds}{\delta_1} + \int_L \frac{ds}{\delta_2} \right)$$

$$\Theta = \frac{M_s}{4F^2G} \left( \frac{a + \pi R + a}{\delta_1} + \frac{a + 4R + a}{\delta_2} \right)$$

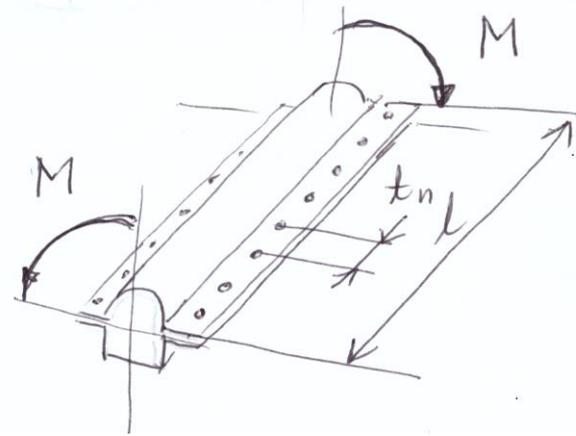
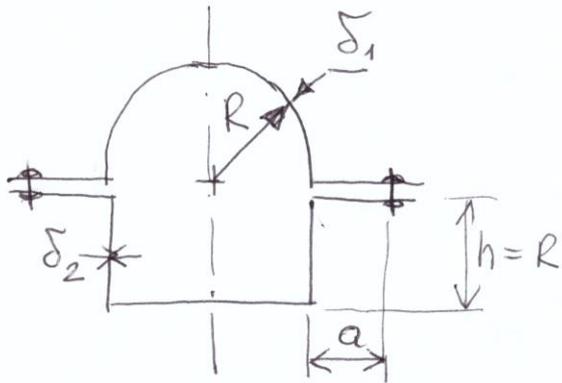
$$\Theta = \frac{10 \cdot 10^3}{4 \cdot 0.0357^2 \cdot 2.6 \cdot 10^{10}} \left( \frac{20 + \pi \cdot 100 + 20}{1.5} + \frac{20 + 4 \cdot 100 + 20}{2} \right)$$

$$\Theta = 7.544 \cdot 10^{-5} (236 + 220) = 0.0344 \frac{\text{rad}}{\text{m}} = 1.97 \text{‰}$$

$$\varphi = \int_0^x \Theta dx$$

$$\varphi(l) = \Theta \cdot l = 1.97^\circ \leftarrow \text{całkowity kąt skręcenia}$$

Rura cienkościenne



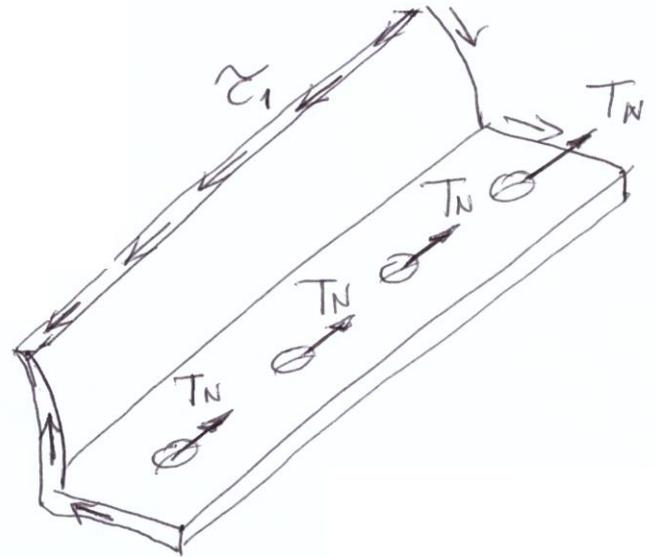
Sily w nitoch  $T_N$  ?

$$\sigma_1 \cdot \delta_1 \cdot l = T_N \cdot n$$

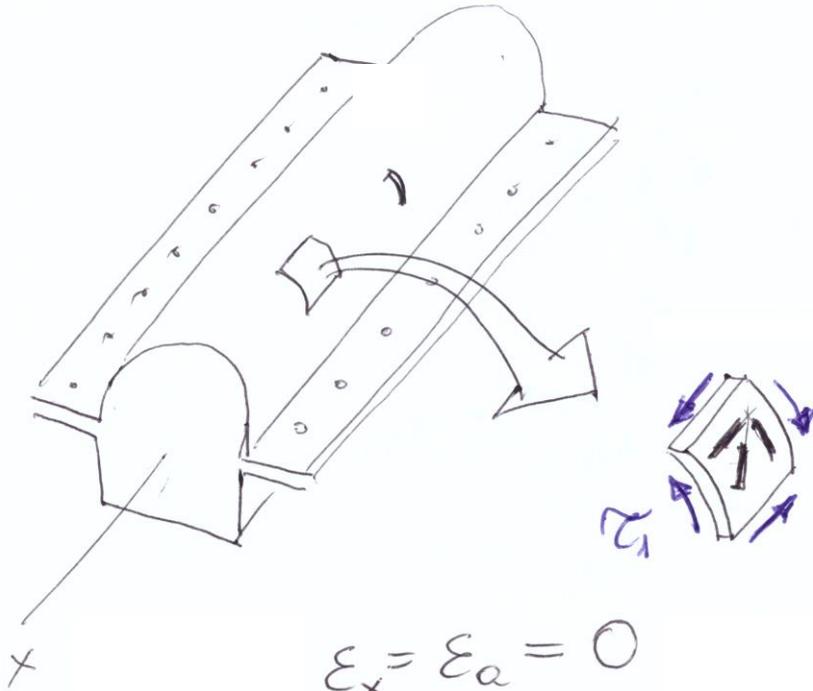
$$T_N = \sigma_1 \cdot \delta_1 \cdot \frac{l}{n}$$

$$T_N = \sigma_1 \cdot \delta_1 \cdot t_n$$

$$T_N = 93 \cdot 1.5 \cdot 20 = 2790 \text{ N}$$



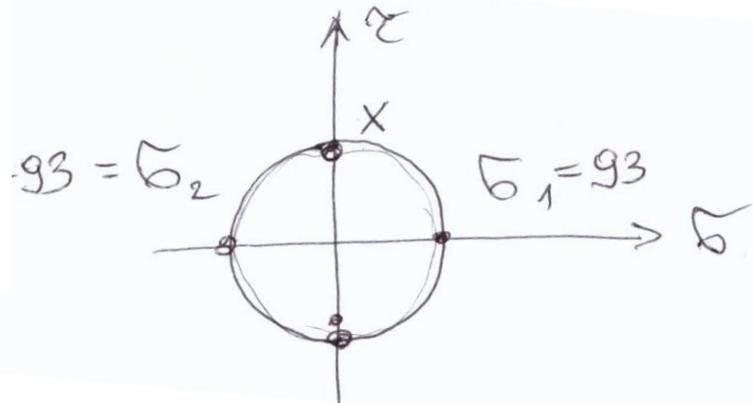
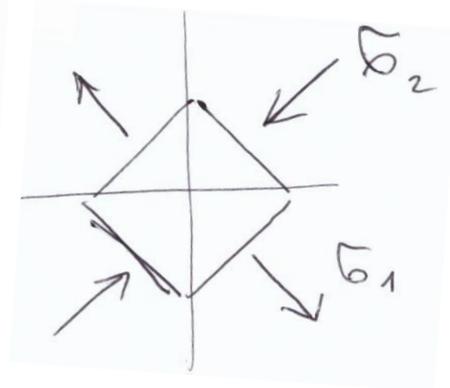
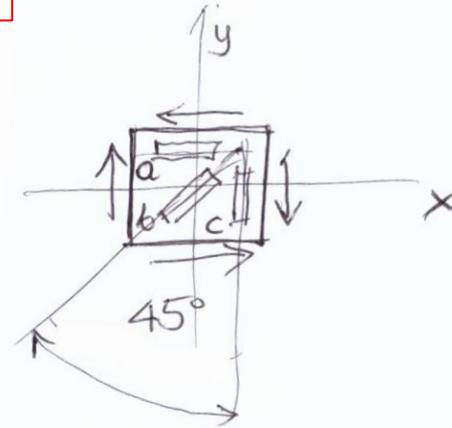
# Wskazania tensometrów



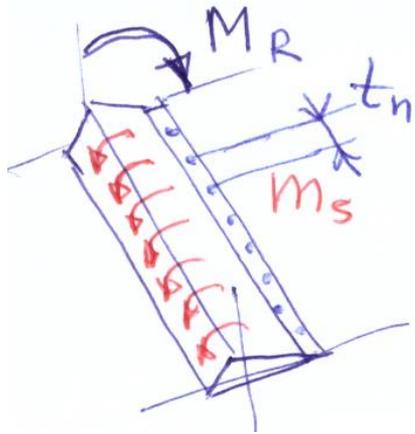
$$\varepsilon_x = \varepsilon_a = 0$$

$$\varepsilon_y = \varepsilon_c = 0$$

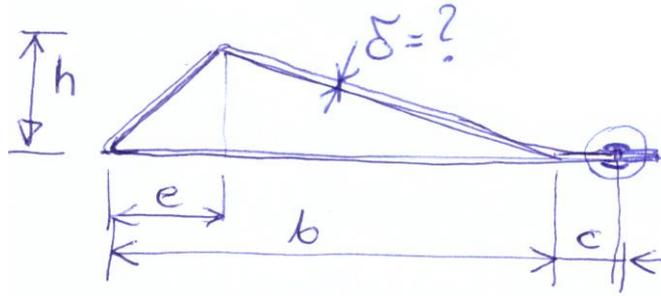
$$\varepsilon_2 = \varepsilon_b = \frac{1}{E} (\sigma_2 - \nu \sigma_1)$$



## Zadanie 7.4 Skręcanie rury cienkościennej



Lotka skrzycone



$$l = 3 \text{ m}$$

$$h = 100 \text{ mm}$$

$$e = 100 \text{ mm}$$

$$b = 400 \text{ mm}$$

$$c = 20 \text{ mm}$$

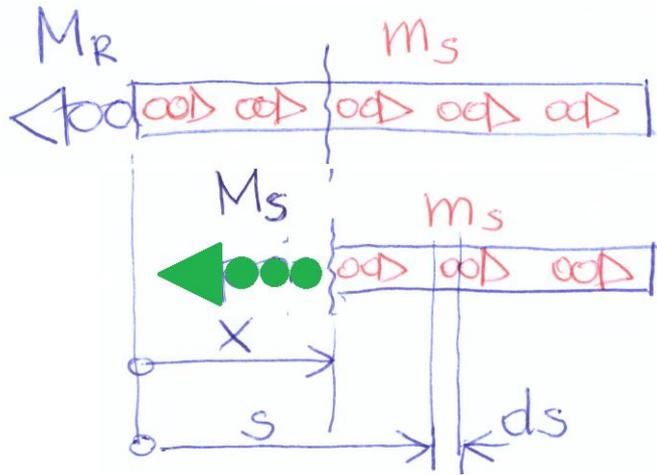
$$t_n = 20 \text{ mm}$$

$$m_s = 2 \frac{\text{kNm}}{\text{m}}$$

$$k_r = 100 \text{ MPa}$$

$$G = 2,6 \cdot 10^4 \text{ MPa}$$

$$\text{Pole: } F = \frac{1}{2} b \cdot h = \frac{0,4 \cdot 0,1}{2} = 0,02 \text{ m}^2$$



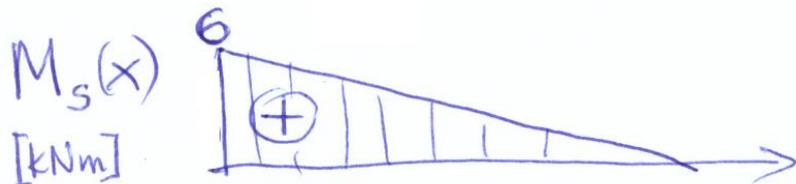
$\tau$ -nie równowagi momentów

$$-M_s + \int_x^l m_s ds = 0$$

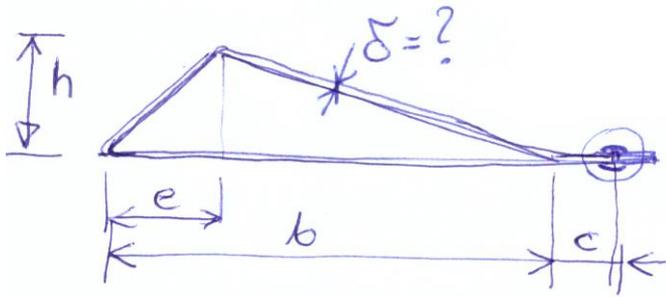
$$M_s(x) = m_s (l - x)$$

$$M_s(0) = m_s \cdot l = 6 \text{ kNm}$$

$$M_s(l) = 0$$



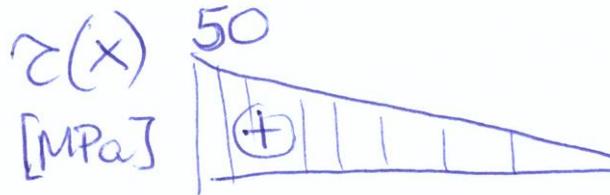
## Zadanie 7.4 (c.d.)



Naprzężenia

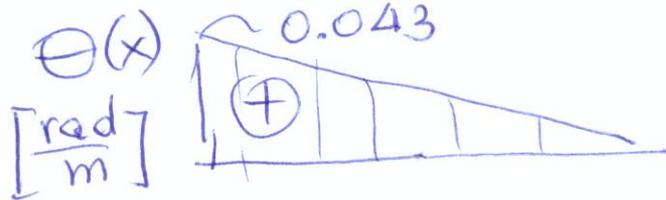
$$\tau = \frac{M_s}{2F\delta} \quad \tau_{\max} = \frac{M_s(0)}{2F\delta}$$

$$\sigma_{\text{red}}^T = 2\tau_{\max} = \frac{M_s(0)}{F\delta} \leq k_r$$

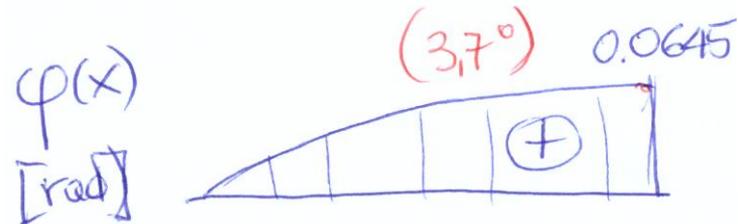


$$\delta \geq \frac{M_s(0)}{F \cdot k_r} = \frac{6000 \text{ Nm}}{0.02 \text{ m}^2 \cdot 100 \cdot 10^6 \frac{\text{N}}{\text{m}^2}} \geq 0.003 \text{ m} = 3 \text{ mm}$$

Jednostkowy kąt skręcenia



$$\Theta = \frac{M_s(x)}{4F^2G} \int \frac{ds}{\delta}$$



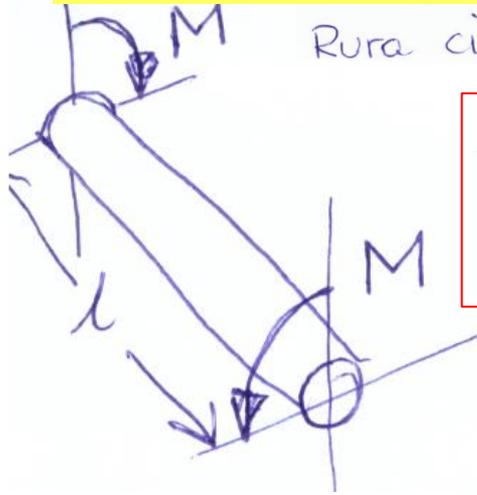
$$\Theta(x) = \frac{M_s(x)}{4 \cdot 0.02^2 \cdot 2.65 \cdot 10^{10}} \cdot \underbrace{\left( \frac{400 + 141 + 316 + 20 + 20}{3} \right)}_{299}$$

$$\Theta(0) = 0.043 \frac{\text{rad}}{\text{m}} = 2.47 \frac{^\circ}{\text{m}}$$

$$\varphi(x) = \int_0^x \Theta(x) dx \quad \varphi(l) = \left[ \frac{0.043 \frac{\text{rad}}{\text{m}} \cdot 3 \text{ m}}{2} \right] = 0.0645 \text{ rad} = 3.7^\circ$$

pole pod krzywą  $\Theta(x)$

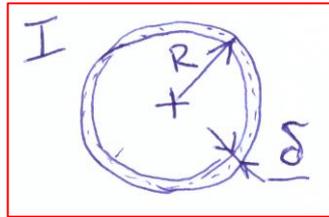
## Zadanie 7.5 Porównanie skręcania rury cienkościennej i pręta otwartego



Rura cienkościenna skręcana momentem  $M$  Dane:  $R = 50 \text{ mm}$

$$\delta = 1 \text{ mm}$$

$$l = 1 \text{ m}$$

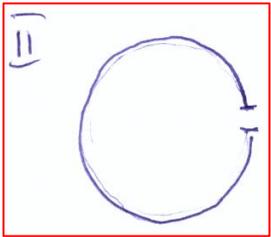


naprężenia tnące

$$\tau_I = \frac{M_s}{2F\delta} = \frac{M_s}{2\pi R^2\delta}$$

jednostkowy kąt skręcenia

$$\Theta_I = \frac{M_s}{4F^2 G} \oint \frac{ds}{\delta} = \frac{M_s}{4\pi^2 R^4 G} \cdot \frac{2\pi R}{\delta} = \frac{M_s}{2\pi G R^3 \delta}$$



Po rozcięciu rury  $J_s = \frac{1}{3} \sum s_i \delta_i^3 = \frac{1}{3} 2\pi R \delta^3$

$$\tau_{II} = \frac{M_s}{J_s} \delta_i = \frac{M_s \cdot \delta}{\frac{2}{3}\pi R \delta^3} = \frac{3}{2} \frac{M_s}{\pi R \delta^2}$$

$$\Theta_{II} = \frac{M_s}{G J_s} = \frac{M_s}{G \frac{2}{3}\pi R \delta^3} = \frac{3}{2} \frac{M_s}{G \pi R \delta^3}$$

Porównanie  
obu prętów

$$\frac{\tau_{II}}{\tau_I} = \frac{3M_s}{2\pi R \delta^2} \cdot \frac{2\pi R^2 \delta}{M_s} = \frac{3R}{\delta} = \boxed{150}$$

$$\frac{\Theta_{II}}{\Theta_I} = \frac{3M_s}{2G \pi R \delta^3} \cdot \frac{2\pi G R^3 \delta}{M_s} = \frac{3R^2}{\delta^2} = \boxed{7500}$$